MCS 452
Solved Problems and Exercises

1) a) What is a norm?

b) What does it mean if a Metric space is complete?

c) Give the definition of a Banach space and give an example.

d) What is the connection between bounded and continuous linear maps?

Bounded linear maps are continuous and all continuous maps are bounded. This is the result of the following theorem:

Thm. Let $T: \mathrm{Dom}\,T \subseteq X \to Y$ be a linear operator, X and Y are normed spaces, then

a) T is continuous $\Leftrightarrow$ T is bounded

b) If T is continuous at a point then it is continuous on whole $\mathrm{Dom}\,T$.

e) What is the dual space of a normed space? (Algebraic dual)

Let X be a vector space. Let us consider the set of all linear functionals $f: X \to \mathbb{K}$. This is a vector space with the addition $(f_1+f_2)(x) = f_1(x)+f_2(x)$ and scalar multiplication $f_1(\alpha x) = \alpha f_1(x)$ $\forall \alpha \in \mathbb{K}$, $x \in X$.

This vector space is called the algebraic dual of X and is denoted by X^*.

2) Let X be a finite dimensional vector space, $\dim X = n$. If $x_0 \in X$ has the property that

$$f(x_0) = 0 \quad (\forall f \in X^*) \quad \text{then} \quad x_0 = 0.$$

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of X and

$$x_0 = \sum_{i=1}^{n} c_i e_i. \text{ Then, } f(x_0) = \sum_{i=1}^{n} c_i f(e_i) = 0 \quad \forall f \in X^*.$$

$$\therefore \; c_i = 0, \; 1 \le i \le n, \; \Rightarrow \; x_0 = 0.$$

3) a) The set of all <u>bounded, linear functionals</u>, denoted by X', is called <u>topological dual</u> (or just dual) of X.

b) Dual space of ℓ^1 is ℓ^∞. (i.e. $(\ell^1)' = \ell^\infty$)
Dual space of c_0 is ℓ^1 (i.e. $(c_0)' = \ell^1$)
Dual space of ℓ^p is ℓ^q with $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$.

4) Define the operator $T : \ell^2 \to \ell^2$ by

$$(Tb)_n = \left(\frac{1}{3}\right)^n b_n \quad \forall n \in \mathbb{N},\ b_n \in \mathbb{R}$$
$$b = (b_1, b_2, \dots) \in \ell^2.$$

a) Show that T is <u>linear</u> on ℓ^2.

b) Show that T is a <u>bounded linear operator</u> on ℓ^2 and determine $\|T\| = ?$

c) Is T invertible?

Take two elements $c, d \in \ell^2$ and $\alpha \in \mathbb{R}$. We have

$$T(c+d) = T(c) + T(d)$$
$$T(\alpha c) = \alpha T(c). \quad \text{So, } T \text{ is } \underline{\text{linear}}.$$

$\| T(b) \|^2 = \left(\frac{1}{3}\right)^2 (b_1)^2 + \left(\frac{1}{3}\right)^4 (b_2)^2 + \dots \leq \left(\frac{1}{3}\right)^2 \| b \|^2$. So,
$\| T(b) \| \leq \frac{1}{3} \| b \|$.

Now taking $p = (1, 0, 0, \dots)$, then $\| T(p) \| = \frac{1}{3} \| p \|$, so
$\| T \| = \frac{1}{3}$.

c) If T is invertible, then $\left(T^{-1}(b)\right)_n = \left(\frac{3}{1}\right)^n (b_n)$. Take $b = (1, \frac{1}{2}, \frac{1}{3}, \dots) \in \ell^2$ and calculate $\| T^{-1}(b) \|$, then we'll see that this norm is <u>not</u> bounded, so $T^{-1}(b) \notin \ell^2$.
i.e. T^{-1} is not exist.

5) Define the functional $F : (C[0,1], \|.\|_\infty) \longrightarrow \mathbb{R}$

$$x \rightarrow F(x) = \int_0^1 t\, x(t)\, dt$$

a) Show that F is a linear functional on $C[0,1]$.

b) Show that F is bounded on $C[0,1]$

c) Choose $x(t) = 1 \;\; \forall\, t \in [0,1]$ and calculate $F(x)$

d) Calculate the norm of F.

a) Take $f, g \in C[0,1]$ and $\alpha \in \mathbb{R}$ and check if

$$F(f+g) = F(f) + F(g) \;,\; F(\alpha f) = \alpha F(f)$$

It is straightforward.

b) $|F(x)| = \left| \int_0^1 t\, x(t)\, dt \right| \leq \int_0^1 |t\, x(t)|\, dt \leq \underbrace{\sup |x(t)|} \cdot \int_0^1 t\, dt$

$$\Rightarrow \quad |F(x)| \leq \|x\|_\infty \cdot \frac{1}{2}$$

$\therefore$ F is bounded.

c) For $x(t) = 1 \;\Rightarrow\; F(1) = \int_0^1 t\, dt = 1/2$

d) From part (b), $\|F\| = \sup_{\|x\|=1} |F(x)| = \boxed{1/2}$.

6)

a) Give a sequence $a = (a_n) \in \ell^2$ but $a \notin \ell^1$

b) Show that $\ell^1 \subseteq \ell^2$.

a) Define $a = (a_n) = \left(\frac{1}{n}\right)$. Then $\int_1^\infty \frac{1}{x}\, dx$ does not exists, but $\int_1^\infty \frac{1}{x^2}\, dx$ exists. $\therefore$ By the integral test, $\sum_{n=1}^\infty \frac{1}{n}$ is divergent.

This implies $(a_n) = \frac{1}{n} \notin \ell^1$ but $(a_n) \in \ell^2$.

b) Choose an arbitrary $x \in \ell^1$. Since $\|x\|_1 = \sum_{n=1}^\infty |x_n| < \infty$, then $\exists\, n_0 \in \mathbb{N}$ s.t. $\forall\, n > n_0$, $|x_n| < 1$. (Because $\lim_{n\to\infty} |x_n| = 0$)

$\therefore$ $|x_n|^2 < |x_n|$ and $\sum_{n=n_0+1}^\infty |x_n|^2 \leq \sum_{n=n_0+1}^\infty |x_n| < \infty \;\Rightarrow\; x \in \ell^2$.

7) Let X be the vector space of <u>all</u> functions f with $f: \mathbb{R} \to \mathbb{R}$. Consider $f_1(x) = 1$, $f_2(x) = \cos^2 x$, $f_3(x) = \cos(2x)$.

a) Prove that f_1, f_2 and f_3 are linearly <u>dependent</u>.

b) Prove that f_2 and f_3 are linearly <u>independent</u>.

a) We can write $f_3(x) = \cos(2x) = 2\cos^2 x - 1 = 2f_2(x) - f_1(x)$

$\therefore$ f_3 is a linear combination of f_1 & f_2.

This set is linearly dependent.

b) $c_1 f_2 + c_2 f_3 = c_1 \cos^2 x + c_2 \cos(2x) = 0 \overset{?}{\Rightarrow} c_1 = 0, c_2 = 0$.

Take $x = \frac{\pi}{2} \Rightarrow c_2 = 0$ and for $x = 0 \Rightarrow c_1 = 0$.

$\therefore$ $c_1 = c_2 = 0$ is the only solution. Therefore f_2 & f_3 are lin. indep.

8) Give statements of the "Open Mapping theorem" and the "closed graph theorem".

9) Let f be a function defined on $[0,1]$. Define

$$g(x) = f(x^2).$$

Show that if $f \in L^3[0,1]$ then $g \in L^1[0,1]$.

$$\|g\|_1 = \int_0^1 |g(x)|\,dx = \int_0^1 |f(x^2)|\,dx \underset{u=x^2}{=} \frac{1}{2}\int_0^1 |f(u)| \cdot \frac{du}{\sqrt{u}}$$

$$\underset{\text{Holder's inequality}}{\leq} \frac{1}{2}\left(\int_0^1 |f(u)|^3 du\right)^{1/3} \cdot \left(\int_0^1 \frac{du}{u^{3/4}}\right)^{2/3} = \sqrt[3]{2} \cdot \|f\|_3 .$$

$\frac{1}{p} = \frac{1}{3}$, $\frac{1}{q} = \frac{2}{3}$. That is $p = 3$, $q = 3/2$.

10) Let $g \in C[0,1]$. Define $T: C[0,1] \to C[0,1]$

$$(Tf)(x) = g(x) + \int_0^x f(x-t)\, e^{-t^2}\, dt.$$

Show that T is a contraction.

By definition $\quad d_\infty(Tf_1, Tf_2) = \sup\limits_{x\in[0,1]} |Tf_1(x) - Tf_2(x)|$

$$= \sup_{x\in[0,1]} \left| \left(g(x) + \int_0^x f_1(x-t)\, e^{-t^2} dt\right) - \left(g(x) + \int_0^x f_2(x-t)\, e^{-t^2} dt\right) \right|$$

$$\leq \sup_{x\in[0,1]} \int_0^x |f_1(x-t) - f_2(x-t)|\, e^{-t^2} dt$$

$$\leq \int_0^x \sup |f_1(x-t) - f_2(x-t)|\, e^{-t^2} dt \leq d_\infty(f_1,f_2) \cdot \int_0^1 e^{-t^2} dt$$

Since $\int_0^1 e^{-t^2} dt \simeq \underbrace{0.747}_{\lambda} < 1$, then T is a contraction.

) For $a \in \mathbb{R}$ and $b > 0$, define $T: C[a,b] \to C[a,b]$

$$(Tf)(x) = a + \int_0^x f(t)\, x e^{-xt} dt.$$

Show that T is a contraction. Therefore, $\exists$ a unique solution for $f \in C[0,\infty)$ to the integral equation $f(x) = a + \int_0^x f(t)\, x e^{-xt} dt$.

We need to show that $d_\infty(Tf_1, Tf_2) \leq \lambda\, d_\infty(f_1,f_2)$ for some $\lambda \in (0,1)$.

et $x \in [0,b]$. Then,

$$d_\infty(Tf_1, Tf_2) = \sup_{x\in[a,b]} |Tf_1(x) - Tf_2(x)|$$

$$= \sup \left| \int_0^x (f_1(t) - f_2(t))\, x e^{-xt} dt \right| \leq \int_0^x \underbrace{\sup |f_1(t) - f_2(t)|}\, x e^{-xt} dt$$

$$= \sup \int_0^x d_\infty(f_1,f_2)\, x e^{-xt} dt \leq d_\infty(f_1,f_2) \cdot \sup_{x\in[a,b]} \underbrace{\int_0^x x e^{-xt} dt}_{1-e^{-x^2}}$$

$$= d_\infty(f_1,f_2) \cdot (\underbrace{1-e^{-b^2}}_{\lambda<1}) = \lambda \cdot d_\infty(f_1,f_2),$$ then T is a contraction.

12) a) State Hölder's inequality for ℓ^p sequence spaces

b) Show that if $\sum_{i=1}^{\infty} |x_i|^{6/5} < 2$ and $\sum_{i=1}^{\infty} |y_i|^6 < 2$ then $\sum_{i=1}^{\infty} |x_i y_i| < 2$

$\ell^p = \{x = (x_i) : \sum_{i=1}^{\infty} |x_i|^p < \infty\}$ $(p \geq 1)$

$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{1/q}$, $\frac{1}{p} + \frac{1}{q} = 1$ (Hölder's Inequality)

$\therefore$ If $p = \frac{6}{5}$, then $\frac{1}{q} = 1 - \frac{1}{p} = 1 - \frac{5}{6} = \frac{1}{6} \Rightarrow q = 6$.

Thus,
$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^{6/5}\right)^{5/6} \cdot \left(\sum_{i=1}^{\infty} |y_i|^6\right)^{1/6} \leq 2^{5/6} \cdot 2^{1/6} = 2.$$

13) a) Define the norm of ℓ^2. Evaluate $\|x\|_2$, where $x = \left(\frac{1}{3^n}\right) \in \ell^2$.

$$\|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2} = \left(\sum_{n=1}^{\infty} \frac{1}{3^{2n}}\right)^{1/2} \leftarrow \text{Geometric series}$$

$a = \frac{1}{3^2}$, $r = \frac{1}{3^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{3^{2n}} = \frac{a}{1-r} = \frac{1}{\cancel{3^2}} \cdot \frac{\cancel{3^2}}{3^2 - 1} = \frac{1}{8}$. $\therefore$ $\|x\|_2 = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$.

b) Give an example of sequence $x = (x_n)$ such that $x \in \ell^2$ but $x \notin \ell^1$!

Let $x = (x_n) = \frac{1}{n}$. Then since $\sum_{n=1}^{\infty} |x_n|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \Rightarrow x = (x_n) \in \ell^2$
($p = 2$, p-series)

But $\sum_{n=1}^{\infty} \frac{1}{n} \leftarrow$ divergent.

Therefore $x = (x_n) \notin \ell^1$.

14) Let $(X, \langle . \rangle)$ be an inner product space and $x, y \in X$. Prove that
$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$, where $\|x\|^2 = \langle x, x \rangle$.

Prove that ℓ^p-space is <u>not</u> an inner product space if $p \neq 2$.