MCS 452 REVIEW PROBLEMS

2) Choose ALWAYS or SOMETIMES or NEVER .
(a) An inner product space is a vector space.
(b) A normed space is an inner product space.
(c) vector space is an inner product space.
(d) An inner product space is a normed space.
(e) A normed space is a vector space.
(f) A vector space is a normed space.
Solutions.
a)ALWAYS b)SOMETIMES c) SOMETIMES d)ALWAYS e)ALWAYS f) SOMETIMES
2) a) Give the statement and proof of the Cauchy- Schwarz Inequality in an Inner Product Space.
Solution.
Let X be an Inner Product Space with inner product $< .;. >$, for every $x, y \in X$ holds that

$$| < x, y > | \leq ||x||||y||$$

3) Prove that if X is an inner product space, then is the inner product $< .;. >: X \times X \to \mathbb{K}$ continuous. This means that if
$x_n \to x$ and $y_n \to y$ then $< x_n, y_n > \to < x, y >$ for $n \to \infty$
Solution.
With the triangle inequality and the inequality of Cauchy-Schwarz

$$| < x_n, y_n > - < x, y > | = | < x_n, y_n > - < x_n, y > + < x_n, y > - < x, y > |$$

$$| < x_n, y_n - y > - < x_n - x, y > | \leq | < x_n, y_n - y > + < x_n - x, y > - < x, y > |$$

$$|||x_n||||y_n - y|| + ||x_n - x||||y|| \to 0$$

since $||x_n - x|| \to 0$ and $||y_n - y|| \to 0$ as $n \to \infty$

4) Prove that
a) if X is an inner product space and A is a non-empty subset of X, then $A^\perp$ is a closed subspace of X.
b) If $A \subseteq B$ then $B^\perp \subseteq A^\perp$
Solution.
Let $x, y \in A^\perp$ and $\alpha \in \mathbb{K}$, then for every $z \in A$,

$$| < x + \alpha y, z > = < x, z > + \alpha < y, z > = 0$$

Thus $A^\perp$ is a vector subspace of X.
Remains to prove $A^\perp = \overline{A^\perp}$.
We already know that $A^\perp \subseteq \overline{A^\perp}$.

For the converse, let $x \in \overline{A^{\perp}}$ then there exist a sequence $\{x_n\}$ in $A^{\perp}$ such that $x_n \to x$. Hence

$$< x, z >= \lim_{n\to\infty} < x_n, z >= 0$$

for every $z \in A$. (Inner product is continuous). So $x \in A^{\perp}$ and $A^{\perp} = \overline{A^{\perp}}$.
b) If $x \in B^{\perp}$ then $< x, y >= 0$ for each $y \in B$ and in particular for every $x \in A \subseteq B$. So $x \in A^{\perp}$ and this gives $B^{\perp} \subseteq A^{\perp}$.
5) Prove that, if X is an inner product space and $S = \{x_1, x_2, \ldots, x_n\}$ is an orthogonal system then S is orthogonal.
Proof. It is already done in class.
6) Let $S = \{x_1, x_2, \ldots, x_n\}$ be an orthonormal set in X and $0 \notin S$ then

$$||x - y|| = \sqrt{2}$$

for every $x \neq y$ in S.
Solution. since S is orthonormal, then for $x \neq y$

$$||x - y||^2 =< x - y, x - y >=< x, x > + < y, y >= 2$$

where $x \neq 0$ and $y \neq 0$ because $0 \notin S$.

7) a) Give the definition of the sesquilinear form.
b) Give the statement of the Riesz Representation Theorem.
c) What means "Hilbert space"? Give an example. (Hint: Look at to the lecture notes.)
8) Consider the Hilbert space $L^2[0;\infty)$ of square integrable real-valued functions, with the standard inner product

$$< f, g >= \int_0^{\infty} f(x)g(x)dx = \lim_{R\to\infty} \int_0^R f(x)g(x)dx$$

Define the linear operator $T : L^2[0;\infty) \to L^2[0;\infty)$ by $(Tf)(x) = f(\frac{x}{5})$ where $f \in L^2[0;\infty)$ and $x \in [0,\infty)$.
a) Calculate the Hilbert-adjoint operator T^*
(Note that $< Tf, g >=< f, T^*(g) >$)
b)Calculate the norm of $||T^*(g)||$ for all $g \in L^2[0;\infty)$ with $||g|| = 1$.
c) Calculate $||T||$.
Solutions. a) $< Tf, g >= \lim_{R\to\infty} \int_0^R f(\frac{x}{5})g(x)dx = \lim_{R\to\infty} \int_0^{R/5} f(y)g(5y)5dy =< f, T^* >$, so $T^*g(x) = 5g(5x)$.
b) $||T*(g)||^2 = \lim_{R\to\infty} \int_0^R |5g(5x)|^2 dx$ so $||T*(g)||^2 = 25 \lim_{R\to\infty} \int_0^{5R} |\frac{1}{5}|g(y)|^2 dy = 5||g||^2$ and this gives that $||T^*|| = \sqrt{5}$
c) $||T|| = ||T^*||$.
9) Let $A : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Define the operator $T : L^2[0;\infty) \to L^2[0;\infty)$ by

$$(Tf)(t) = A(t)f(t)$$

a)Prove that T is a linear operator on $L^2[a;b]$.
b) Prove that T is a bounded linear operator on$L^2[a;b]$.
Solutions.
a) Let $f, g \in L^2[a;b]$ and $\alpha \in \mathbb{R}$ then

$$T(f+g)(t) = A(t)(f+g)(t) = A(t)f(t) + A(t)g(t) = T(f)(t) + T(g)(t)$$

and

$$T(\alpha f)(t) = A(t)(\alpha f)(t) = \alpha A(t)(f)(t) = \alpha T(f)(t)$$

b) $||(Tf)|| \leq ||A||_\infty ||f||$ with $||.||_\infty$ the sup-norm.
A is continuous and because $[a,b]$ is bounded and closed, then $A||_\infty = \max_{t\in[a,b]} |A(t)|$.
10) Consider the space $C[0,1]$ with an inner product $< f, g >= \int_0^1 f(t)g(t)dt$.
Let $f_1(t) = t^2 + 1$ and $f_2(t) = 1 + t$. Use Gram-Schmidt process to find an orthogonal basis for $Span\{f_1, f_2\}$.
Solution. Begin by letting $g_1 = f_1$ and then define

$$g_2 = f_2 - \frac{< f_2, g_1 >}{< g_1, g_1 >} g_1$$

and evaluate the inner products.
11) In $\mathbb{R}^3$, let $u = (0,1,0)$ and $v = (1,-1,2)$. Define the inner product by

$$< u, v >= Au.Av = u^T.A^T.A.v$$

where A is the 3×3 matrix with rows $[1,0,2],[0,3,-1],[1,0,1]$,
a) Compute $< u, v >$
b) Compute the norms $||u||, ||v||$
c) Find all vectors w perpendicular to u
(Hint. a similar question is solved in the lecture)